Math 10B with Professor Stankova
Worksheet, Discussion #12; Tuesday, 3/5/2019
GSI name: Roy Zhao

1 Random Variables

1.1 Concepts

1. A **random variable** is any function $X : \Omega \to \mathbb{R}$. It isolates some concept that we care about. For example, when we flip a coin 20 times, then we can define a random variable which is the number of heads that we flip.

   A **probability mass function (PMF)** is a function from $\mathbb{R}$ to $[0, 1]$ that is associated to a random variable X. We define $f(x) = P(X = x) = P(X^{-1}(\{x\}))$.

   Two random variables X, Y are called **independent** if for any subsets $E, F \subset \mathbb{R}$, the subsets $X^{-1}(E), Y^{-1}(F) \subset \Omega$ are independent. To prove that two random variables are independent, we need to show that those two sets are independent for any two choices of E, F (actually, it suffices to only consider E, F as one point sets or that $P(X = x, Y = y) = P(X = x)P(Y = y)$ for any $x, y \in \mathbb{R}$). To prove that they are not independent, we only need to find one counterexample pair E, F.

1.2 Examples

2. Suppose that we roll two die and let X be equal to the maximum of the two rolls. Find $P(X \in \{1, 3, 5\})$ and draw the PMF for X.

Solution: First we draw the PMF. We calculate $P(X = x)$ by counting the number of ways we can roll two die so that the maximum is x and then dividing by the total number of possibilities, which is 36. So for instance, the only way to get $X = 1$ is if we roll $(1, 1)$ and hence $P(X = 1) = \frac{1}{36}$. Then $P(X = 2) = \{(1, 2), (2, 2), (2, 1)\}/36 = \frac{3}{36}$. Thus, we have that

$$f(1) = \frac{1}{36}, f(2) = \frac{3}{36}, f(3) = \frac{5}{36}, f(4) = \frac{7}{36}, f(5) = \frac{9}{36}, f(6) = \frac{11}{36}.$$

We draw the PMF with stalks at 1 through 6 of those respective heights. Then $P(X \in \{1, 3, 5\}) = P(X = 1) + P(X = 3) + P(X = 5) = \frac{1}{36} + \frac{5}{36} + \frac{9}{36} = \frac{15}{36} = \frac{5}{12}$.

3. When rolling two die, let Y be equal to the first die roll. Are X, Y independent random variables?

Solution: No. Intuitively, if we know that the first die roll is a 6, then the maximum has to be a 6. Mathematically writing that, we see that $P(X = 6, Y = 6) = P(Y = 6)$ and $P(X = 6) \neq 1$ so $P(X = 6, Y = 6) \neq P(X = 6)P(Y = 6)$.

1.3 Problems

4. True **FALSE** A RV goes from subsets of Ω to $\mathbb{R}$.

Solution: A RV goes from Ω unlike the probability function, which goes from subsets of Ω.

5. True **FALSE** Similar to the probability function, a PMF takes events or subsets of $\mathbb{R}$ and assigns a probability between $[0, 1]$.

Solution: The PMF takes single values in $\mathbb{R}$ and assigns a probability between $[0, 1]$. This is one reason that we can draw the PMF.

6. I flip a fair coin 4 times. Let X be the number of heads I get. Draw the PMF for X.

Solution: The range is $\{0, 1, 2, 3, 4\}$. Then $P(X = x)$ is the number of ways to get x heads over the total number of ways so $P(X = x) = \frac{\binom{4}{x}}{2^4}$.

7. I roll two fair four sided die with sides numbered $1 - 4$. Let X be the product of the two numbers rolled. Find the range of X and draw the PMF for X.

Solution: The range is all products of two numbers in $\{1, 2, 3, 4\}$. This is $\{1, 2, 3, 4, 6, 8, 9, 12, 16\}$. We calculate:

x	1	2	3	4	6	8	9	12	16
$f(x)$	$\frac{1}{16}$	$\frac{2}{16}$	$\frac{2}{16}$	$\frac{3}{16}$	$\frac{2}{16}$	$\frac{2}{16}$	$\frac{1}{16}$	$\frac{2}{16}$	$\frac{1}{16}$

8. (Challenge) I draw 5 cards from a deck of cards. Let X be the number of hearts I draw. What is the range of X and draw the PMF of X. Use this to find the probability that I draw at least 2 hearts.

Solution: The range is $\{0, 1, 2, 3, 4, 5\}$. To calculate $f(x) = P(X = x)$, we count the number of good ways over the total number of ways. The number of good ways to draw x hearts is to first pick out x hearts out of the 13 hearts, and then fill out the rest of the hand and pick $5 - x$ non-heart cards from the remaining 39 cards. Thus $f(x) = \frac{\binom{13}{x}\binom{39}{5-x}}{\binom{52}{5}}$. Thus, we have that $P(X \geq 2) = P(X = 2) + P(X = 3) + P(X = 4) + P(X = 5) = \frac{\binom{13}{2}\binom{39}{3}+\binom{13}{3}\binom{39}{2}+\binom{13}{4}\binom{39}{1}+\binom{13}{5}\binom{39}{0}}{\binom{52}{5}}$.